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# AN INTEGRAL EQUATION OCCURRING IN A PROBLEM OF SUBSIDENCE

J. H. GIESE





Department of the Army Project No. 503-06-002
Ordnance Management Structure Code No. 5010.11.812

BALLISTIC RESEARCH LABORATORIES

ABERDEEN PROVING GROUND, MARYLAND

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An integral equation occurring in a problem of subsidence

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JHGiese/bjw
Aberdeen Proving Ground, Md.
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# AN INTEGRAL EQUATION OCCURRING IN A PROBLEM OF SUBSIDENCE

ABSTRACT

The linear boundary value problem that was solved by numerical methods in BRL Report No. 1131 has been reformulated as a linear integral equation and solved by Laplace transform methods.

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## INTRODUCTION

As a result of mining operations cavities are produced into which the overburden may eventually collapse. Following collapse there will be a period during which the fallen material subsides. It would be extremely difficult to treat this motion in a way that takes into account all the geometrical and physical peculiarities of a particular situation. However, Leser and Jenike[1] have proposed a semi-empirical idealized one-dimensional model of subsidence which is based on the following system of linear partial differential equations.

$$\partial u/\partial T = -\alpha \partial y/\partial T \tag{1.1}$$

$$\partial y/\partial t = u \tag{1.2}$$

Here $y(t,T)$ is the height above the top surface of the cavity at time t of the layer of material through which the plane of disturbance passed at time T; $u(t,T)$ is the velocity of this layer; and

$$\alpha(t,T) = K \left[1 + K(t - T)\right]^{-1} - \left[1 + t - T\right]^{-1} \tag{1.3}$$

is an empirical function, in which the constant $K > 1$. By eliminating $y(t,T)$ between (1.1) one obtains for $u(t,T)$ the linear hyperbolic partial differential equation

$$\frac{\partial^2 u}{\partial t \partial T} = \frac{-2K}{1 + K(t-T)} \frac{\partial u}{\partial T} \tag{1.4}$$

Boundary conditions for (1.4) were determined by the following considerations. Let $\gamma_s$ be the original density of the undisturbed overburden, and $\gamma_o \quad \gamma_s$ the density immediately after passage of the disturbance, both being assumed constant. Then conservation of matter at the wave front $y(t,t)$ implies

$$\gamma_s \, dy(T, T)/dT = \gamma_o \left[dy(T, T)/dT - u(T, T)\right] \tag{1.5}$$

By (1.1) - (1.3) this can be rewritten as

$$\partial u(T, T)/\partial T = \gamma_s (K - 1) (\gamma_s - \gamma_o)^{-1} u(T, T) \tag{1.6}$$

On $T = 0$ it was assumed on empirical grounds that for some constant $\epsilon > 0$

$$u(t, 0) = -\epsilon\gamma_s (K - 1)(\gamma_s - \gamma_o)^{-1}(1 + Kt)^{-2} \qquad (1.7)$$

Leser and Jenike solved this problem originally by numerical integration. However, as we shall show in the following sections:

(i) An integral representation (2.6) can be found for $u(t, T)$ in terms of $u(t, t)$;

(ii) $u(t, t)$ satisfies a Volterra integral equation (2.8) of the second kind, of convolution type, that depends on the single parameter

$$a = (1 - K^{-1})(1 - \gamma_o\gamma_s^{-1})^{-1} > 0 \qquad (1.8)$$

(iii) $\lim_{t=\infty} u(t, t) = 0$ or $\infty$ as $a \leq 1$ or $a > 1$.

(iv) $1 \leq (1 + t)^2 u(t, t) \leq m^{-2}(a)$ for some $m(a)$, $a < 1$.

(v) $u(t, t)$ is a strictly increasing function of <u>a</u> for fixed t.

(vi) $u(t, t)$ is a strictly decreasing function of t for fixed <u>a</u> such that $0 < a \leq 1$.

(vii) The wave-front $y(t, t)$ advances only a finite distance if $a < 1$.

The author wishes to acknowledge his indebtedness to Dr. T. Leser for calling his attention to this problem, and to Dr. W. C. Taylor for suggestions concerning its treatment by Laplace transform theory.

REFORMULATION-REDUCTION TO AN INTEGRAL EQUATION

If we let

$$w(t, T) = -\frac{(\gamma_s - \gamma_o)}{\epsilon\gamma_s(K - 1)} u\left(\frac{t}{K}, \frac{T}{K}\right) \qquad (2.1)$$

then (1.5) - (1.7) become

$$\frac{\partial^2 w}{\partial t \partial T} = -\frac{2}{1 + t - T} \frac{\partial w}{\partial T} \qquad (2.2)$$

$$\frac{\partial w(T,T)}{\partial T} = a\, w\,(T,\ T) \tag{2.3}$$

$$w(t,\ 0) = (1 + t)^{-2} \tag{2.4}$$

where

$$a = \gamma_s\ (K - 1)/K\ (\gamma_s - \gamma_o) > 0 \tag{2.5}$$

Now (2.2) to (2.4) imply

$$w(t,\ T) = \frac{1}{(1 + t)^2} + a \int_0^T \frac{w(r,r)\ dr}{(1 + t - r)^2} \tag{2.6}$$

and if we let

$$v(t) = w(t,\ t) \tag{2.7}$$

then by (2.6)

$$v(t) = \frac{1}{(1 + t)^2} + a \int_0^t \frac{v(r)dr}{(1 + t - r)^2} \tag{2.8}$$

Thus $v(t)$ is defined as a solution of a Volterra integral equation of the second kind of convolution type. The standard theorem for such equations [2] asserts the existence of $v(t)$, and in fact yields

$$(1 + t)^{-2} \leqq v(t) \leqq e^{at}\ (1 + t)^{-2} \tag{2.9}$$

Also by (2.6)

$$(1 + t)^{-2} \leqq w(t,\ T) \leqq w(t,\ t) = v(t) \tag{2.10}$$

and, actually, $w(t,\ T)$ is a strictly increasing function of $T$ for fixed $t = T$. Since the positive function $w(t,\ T)$ is dominated by $v(t)$, in order to provide a basis for estimating $w$ it will suffice to consider only $v(t)$.

## APPLICATION OF LAPLACE TRANSFORMS

By (2.9) $v(t)$ is of exponential type. Accordingly its Laplace transform

$$V(s) = \int_0^{\infty} e^{-st} v(t)\, dt \tag{3.1}$$

is a regular function of s for Re $s > a$. Also

$$f(t) = (1 + t)^{-2} \tag{3.2}$$

is bounded, and its Laplace transform

$$F(s) = \int_0^{\infty} e^{-st} f(t)\, dt \tag{3.3}$$

is regular for Re $s > 0$. Since

$$(e^{-s}F(s))'' = e^{-s}/s \tag{3.4}$$

and since

$$e^{-s}F = (e^{-s}F)' = 0 \text{ for Re } s = \infty,$$

then (3.4) implies, after an integration by parts, that

$$F(s) = 1 - se^{s} \int_s^{\infty} r^{-1} e^{-r}\, dr \tag{3.5}$$

which is regular in the entire s -plane except for a slit along the negative real-axis.

In the region where both $V(s)$ and $F(s)$ exist the Laplace transform of (2.8)

$$V(s) = F(s) + aF(s)V(s)$$

and then (3.5) and

$$V(s) = F(s)/\left[1 - a F(s)\right] \tag{3.6}$$

define the analytic continuation of $V(s)$ into the slit s-plane. To determine the nature and location of the singularities of V, first consider F. If we move the point s around a closed path that contains $s = 0$ we obtain from (3.5)

$$F(se^{2\pi i}) = F(s) + 2\pi i s e^{s}$$

Accordingly

$$F(s) = H(s) + se^{s}\log s \tag{3.7}$$

where $H(s)$ is an entire function. Thus $F(s)$ has a <u>logarithmic branch point</u> at $s = 0$. Since F is real on the positive real axis, $H(s)$ is real on the entire real axis. Now

$$\operatorname{Im} F(s) = \begin{cases} 0 & \text{for } s \geqq 0 \\ \pm se^{s} & \text{for } s \leqq 0 \end{cases}$$

where the upper (lower) sign refers to the upper (lower) side of the slit. By the maximum principle for harmonic functions $\operatorname{Im} F(s)$ is negative (positive) on the upper (lower) half of the s-plane. Consequently the roots of

$$1 - aF(s_o) = 0 \tag{3.8}$$

can lie only on the positive real axis or at $s = 0$. Since $F(0) = 1$, and since for $s \geqq 0$

$$F'(s) = -\int_0^\infty e^{-st}\, tf(t)dt < 0$$

(3.8) has a single simple root $s_o > 0$ if and only if $a > 1$; and $s_o = 0$ if and only if $a = 1$. Thus, if $a > 1$, then $V(s)$ has a single <u>simple pole</u> at $s_o$ in addition to a branch point at 0; if $0 < a < 1$ it has only a branch point.

The complex inversion formula yields

$$v(t) = (2\pi i)^{-1} \int_{x-i\infty}^{x+i\infty} e^{ts}\, V(s)\, ds \qquad (3.9)$$

where $x > s_o$. An estimate that can be used to bound $|V(s)|$ can be found by integrating (4.5) twice by parts to obtain

$$F(s) = s^{-1} - 2s^{-2} + 6se^{s} \int_{s}^{\infty} r^{-4} e^{-r}\, dr$$

If the path of integration of F(s) is taken to be the half-line $|s| \leqq r \leqq \infty$ and the circular arc $r = |s|\, e^{i\theta}$, $0 < |\theta| \leqq \phi \leqq \pi$, then

$$|F(s) - s^{-1} + 2s^{-2}| \leqq 6\, |s|^{-3} e^{-2|s| \sin^2\phi/2} + 6\, |s|^{-2} \qquad (3.10)$$

By standard techniques and application of (3.10) the path of integration in (3.9) can be deformed into a circle C about $s_o$ (if $a > 1$) and a path that traverses both sides of the negative real axis. The circle contributes

$$(2\pi i)^{-1} \int_C \frac{e^{st} F(s)\, ds}{1 - a\, F(s)} = - e^{s_o t} F(s_o)/aF'(s_o), \quad a > 1 \qquad (3.11)$$

where, by (3.5) and (3.9)

$$F(s_o) = 1/a, \quad aF'(s_o) = (1 + s_o^{-1})(1 - a) + a \qquad (3.12)$$

The negative real axis contributes

$$I(t) = (2\pi i)^{-1} \int_{-\infty}^{0} e^{tx} (V^- - V^+)\, dx$$

where $V^-$ ($V^+$) designates values on the lower (upper) side of the axis and $x = \text{Re } s = s$. Now write

$$V^- - V^+ = F^- /(1-aF^-) - F^+/(1-aF^+) = (F^- - F^+)/(1-aF^-)(1-aF^+)$$

By (3.7) $F^+ = \overline{F^-}$ and $F^- - F^+ = -2\pi i x e^x$, whence

$$I(t) = -\int_{-\infty}^{0} x e^{(1+t)x} \left|1 - aF^-(x)\right|^{-2} dx > 0. \quad (3.13)$$

If $a \neq 1$, then $|1 - aF^-| \neq 0$ on $-\infty \leq x \leq 0$, since $F^-(-\infty) = 0$ and $F^-(0) = 1$. Then if we let

$$m(a) = \min |1 - aF^-| > 0$$

we find

$$0 < I(t) < -m^{-2} \int_{-\infty}^{0} x e^{(1+t)x} dx = m^{-2}(1 + t)^{-2}, \ a \neq 1. \quad (3.14)$$

Now we can prove the assertions of Section 1. If $a > 1$, then

$$v(t) = I(t) - e^{s_0 t} F(s_0)/aF'(s_0)$$

which implies part of (iii). If $a < 1$, then $v(t) = I(t)$, and (3.14) implies another part of (iii). In conjunction with $v = (1 + t)^{-2}$ (3.14) yields (iv).

The case $a = 1$ remains to be treated. First note that by (3.7) $F(0) = 1$ implies $H(0) = 1$. Now let

$$H(s) = 1 + \sum_{1}^{\infty} a_n s^n$$

where the coefficients $a_n$ are real. Then

$$1 - F^- = - \sum_1^{\infty} a_n x^n - xe^x \left[\log(-x) - \pi i\right]$$

and

$$\left|1 - F^-\right|^2 = x^2 \left\{\left[\sum_1^{\infty} a_n x^{n-1} + e^x \log(-x)\right]^2 + \pi^2 e^{2x}\right\} \quad (3.15)$$

The convergence of I at its upper limit follows from

$$- \int_{-b}^{0} \frac{e^{(t-1)x} dx}{x \log^2(-x)} \leq - N \int_{\log b}^{-\infty} u^{-2} du = - N/\log b \quad (3.16)$$

where $0 < b < 1$, and $N = \max(1, e^{(1-t)b})$. The convergence of I at its lower limit follows from $\left|1 - F^-\right|^2 = O(x^2 e^{2x})$ for $x \to -\infty$. Now we can easily deduce

$$\lim_{t=\infty} v(t) = 0 \text{ \underline{if} } a = 1.$$

Proof: Write

$$I = \int_{-\infty}^{-b} + \int_{-b}^{0} x\, e^{(1+t)x} \left|1 - F^-\right|^{-2} dx = I_1 + I_2$$

and suppose $t > 1$. First, choose $b < 1$ so small that $\left|1 - F^-\right|^2 \geq 0.5x^2 \log^2(-x)$ in $I_2$. Then choose $\theta > 0$ and reduce b, if necessary, so that $-2/\log b < \theta$. Then by (3.16) $0 < I_2 < \theta$ for all $t > 1$. Next, if we let

$$\int_{-\infty}^{-b} x \left|1 - F^-\right|^{-2} dx = P$$

then

$$I_1 = e^{-(1+t)b} P,$$

and for

$$t > 1 - b^{-1} \log (\Theta/P)$$

we have $0 < I_1 < \Theta$. Finally, for such $b$ and $t$, $0 < I < 2\Theta$.

The standard interative technique for solving (2.8) would yield a power series in a with coefficients that are positive functions of t. Thus v(t) is a monotone function of a for all $t > 0$. This and the representation $v(t) = I(t)$ for $0 < a \leqq 1$, and (3.13) imply (v) and (vi).

Finally, let us consider the motion of the wave-front $y = y(t, t)$. By (1.5), (2.1), and (2.7)

$$dy(t, t)/dt = \epsilon(K - 1) \gamma_o \gamma_s (\gamma_s - \gamma_o)^{-2} v(K t)$$

and if $y(0, 0) = 0$, then

$$y(t, t) = \epsilon K^{-1}(K - 1) \gamma_o\gamma_s(\gamma_s - \gamma_o)^{-2} \int_0^{Kt} v(s)\, ds \qquad (3.17)$$

By (iv), if $a \neq 1$.

$$1 \leqq (1 + t^{-1}) \int_0^t v(s)\, ds = m^{-2}(a) ,$$

which implies (vii).

J. H. Giese

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